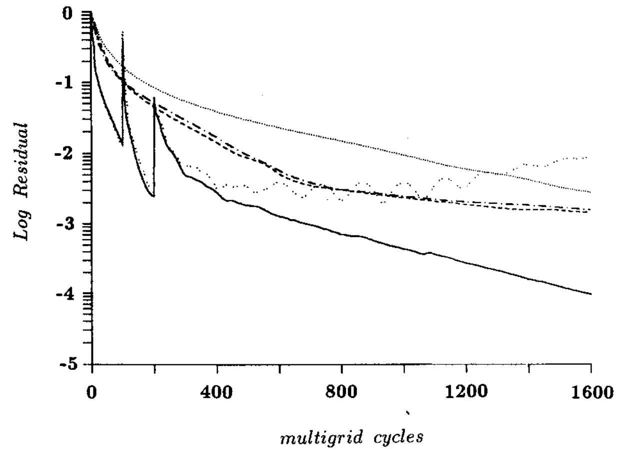

Fig. 3 Strong shock-wave/boundary-layer interaction, influence of smoothing and grid transfer operators on convergence: · · ·, SG unsmoothed; ---, SG SRS; —·—, SG ERS; ·· ·· MG SRS/SCP; —, MG ERS/ECP.

even though the computational overhead per cycle is greater (roughly 45% larger).

The second test case corresponds to the hypersonic (laminar) flow over a compression ramp. This test case has been selected to show the influence of both grid transfer operators and flow complexity on the convergence rate. The flow conditions are those of the previous case and are such that an extended separation region is present due to the strong viscous-inviscid interaction. The geometry is composed of a flat plate of length $L_p = 25$ cm, followed by a 15-deg wedge of length $L_w = 10$ cm. Several computations have been performed, and the convergence rates are shown in Fig. 3. In particular, the convergence of the single grid solution with and without smoothing is compared vs the multigrid with standard grid transfer operators, as well as direction-dependent operators. The results show that in the presence of a large separation extent convergence is mainly driven by the development of the recirculation region. Moreover, the standard multigrid strategy does not converge.

Conclusions

In the present work a multigrid technique for viscous hypersonic flows has been developed. The technique is based on the FAS-FMG method and it uses a V-cycle strategy. Applications to hypersonic flows show that the grid transfer operators play a crucial role for the robustness and effectiveness of the technique: 1) to inhibit upstream propagation of high-frequency disturbances, the prolongation operator must have a directionality character; 2) the robustness of the algorithm is increased by direction-dependent implicit residual smoothing; and 3) in the presence of strong shock-wave boundary-layer interaction the convergence is affected by the development of the recirculation region.

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Boundary Formulations for Sensitivity Analysis Without Matrix Derivatives

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Introduction

A NEW hybrid approach to continuum structural shape sensitivity analysis employing boundary element analysis (BEA) is described that uses iterative reanalysis to obviate the need to factor perturbed matrices in the determination of surface displacement and traction sensitivities via a univariate perturbation/finite difference (UPFD) step. Sensitivities of derived surface stresses and interior displacements and stresses are determined by an analytical formulation using these surface sensitivities. This algorithm is shown to be superior to a related semi-analytical approach. Test cases are discussed, and it is concluded that the technique is viable for shape sensitivity analysis from the software engineering perspective because it avoids computation of derivatives of BEA matrices.

Shape design sensitivity analysis (DSA) denotes computing rates of change of an object's response with respect to changes in design variables that control its shape. Implicit differentiation[1,2] is effective for computing sensitivities of surface displacement and tractions and avoids factoring perturbed matrices. However, it requires derivatives of BEA coefficient matrices. For a general BEA program, the formulation and implementation of this capability is a considerable effort. For sophisticated elements (i.e., trimmed patch elements),[3,4] such activity necessitates computation of sensitivities of element differential geometry, which requires sensitivities of the surface geometric model. The boundary integral equations (BIEs) associated with implicit differentiation are also more complex than those in standard BEA, thus requiring more time in the numerical integration process.

This new technique obviates the need to compute BEA matrix sensitivities in the DSA process by employing a UPFD strategy to determine surface displacement and traction sensitivities in concert with an iterative reanalysis technique.[5] The effect of this strategy is two-fold. The UPFD approach allows the immediate reuse of existing subroutines for computation of BEA matrix coefficients in the DSA process. The reanalysis technique provides for economical response computation of univariately perturbed models without factoring perturbed matrices. A formulation is shown, contrasting this new approach with a semi-analytical technique for DSA presented elsewhere.[6] Examples show that this new approach is equivalent to the more conventional analytic sensitivity analyses and

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superior to semi-analytical approaches in the prediction of surface stress sensitivities. This new approach thus provides substantial computational economy without the burden of a large-scale reprogramming effort.

Continuum Shape Reanalysis

Reanalysis characterizes techniques for subsequent analysis of modified problems using fewer resources than required in the original. An iterative reanalysis boundary technique[5] has been described that allows for the economical prediction of the response of modified continuum shapes. This technique obviates the need to form triangular factors of perturbed BEA matrices, instead relying on the reuse of the factorization of the original (unmodified) matrix. Two schemes were presented. Both approaches solve the algebraic equations associated with the modified BEA model using the $[L][U]$ factorization of the matrix associated with the unmodified model. The equations for the original and modified models are

$$[F]\{u\} = [G]\{t\} \tag{1}$$

and

$$([F] + [\Delta F])(\{u\} + \{\Delta u\}) = ([G] + [\Delta G])(\{t\} + \{\Delta t\}) \tag{2}$$

where $\{u\}$ and $\{t\}$ are column vectors of node point unknown and specified response quantities, and $[F]$ and $[G]$ are coefficient matrices occurring in BEA. The quantities preceded by Δ represent changes in these entities caused by a shape modification. These reanalysis schemes are listed as simple iteration

$$[F] + \{\Delta u^{(k+1)}\} = \{v\} - [\Delta F]\{\Delta u^{(k)}\} \tag{3}$$

where

$$\{v\} = [\Delta G]\{t\} - [\Delta F]\{u\} \tag{4}$$

and scaled, two-step iteration

$$\{\Delta u^{(k+2)}\}_{,L} = \{\Delta u^{(k)}\}_{,L} + \frac{1}{1+\alpha}\left[[I] + \left(\frac{\alpha[I]-[B]}{1+\alpha}\right)\right]\{z\} \tag{5}$$

where

$$\alpha = (b + c)/(a + d) \tag{6}$$

$$\{z\} = \left(\frac{\alpha[I] - [B]}{1+\alpha}\right)^2 \{z\} \tag{7}$$

and

$$\{z\} = [F]^{-1}\{v\} \tag{8}$$

$$[B] = [F]^{-1}[\Delta F] \tag{9}$$

$$\{v^{(1)}\} = [B]\{z\} \tag{10}$$

$$a = \{z\}^T\{z\};\ b = \{z\}^T\{v^{(1)}\};\ c = \{v^{(1)}\}^T\{v^{(1)}\} \tag{11}$$

where α is called the scale factor. The two-step strategy was shown to generate a more efficient reanalysis and extend convergence for more highly modified models. Convergence is dependent on the norm of $[B]$ for simple iteration and the norm of $[M]$ for the two-step approach, where

$$[M] = \left(\frac{\alpha[I]-[B]}{1+\alpha}\right) \tag{12}$$

An issue in the two-step formulation was determining α to produce an $[M]$ with norm less than $[B]$. For such values, convergence of the scaling technique was superior to simple iteration. Two algorithms were demonstrated employing scaling. In the first, α is computed in the first pass of the process and held constant. In the second, a new α was computed in each iteration, using current entries in Eqs. (6) and (11). $[B]$ is used symbolically, but never actually computed or assembled in explicit form. $[B]$ is always postmultiplied by a known vector, and the actual operations performed consist of the matrix multiplication followed by backward substitution using the factorization of the original left-hand-side matrix $[F]$.

Sensitivity Analysis by Implicit Differentiation

Implicit differentiation[1,2] of the algebraic system equations yields

$$\frac{\partial}{\partial X_L}([F]\{u\} = [G]\{t\}) \tag{13}$$

or

$$[F]\{u\}_{,L} = ([G]_{,L}\{t\} - [F]_{,L}\{u\}) = \{r\} \tag{14}$$

Note $\{u\}$ contains displacements at nodes with specified tractions and tractions at nodes where displacements are specified, and X_L represents the Lth design variable. Equation (14) reveals a fundamental characteristic of implicit differentiation. If $\{r\}$ can be formed, $\{u\}_{,L}$ can be determined using the $[L][U]$ factorization of $[F]$ computed in the previous analysis step. This approach relies upon the availability of the derivatives of the BEA $[F]$ and $[G]$ matrices. These matrices can be assembled from contributions from boundary integrals associated with individual element/load point pairs as described in Refs. 1 and 2. This formulation produces a BIE with two term kernel functions comprised of fundamental solutions, Jacobians, and their sensitivities. Implementation of this formulation thus requires considerable work. These BIEs are also more complex than those in standard BEA, requiring more time for numerical integration.

Alternatively, a semi-analytical approach[6] can be used that also exploits the beneficial attribute of implicit differentiation, avoiding matrix factorization. In this approach, matrix sensitivities are obtained via a UPFD approximation at the matrix level as

$$[F]_{,L} = \frac{[\tilde{F}] - [F]}{\Delta X_L}; \qquad [G]_{,L} = \frac{[\tilde{G}] - [G]}{\Delta X_L} \tag{15}$$

where symbols with a tilde represent matrices associated with a shape made from design variables $\{X_1, X_2, \ldots, X_L + \Delta X_L, \ldots\}$. This approach can reuse existing BEA subroutines to compute matrix coefficients in the DSA step. However, this approach is known to be sensitive to perturbation step sizes.

Univariate Perturbation/Finite Difference Technique

Reanalysis techniques have convergence behavior dependent on the amount of shape modification. For small changes, all methods converge in a few iterations. This result suggested a compelling role for reanalysis in the analysis of the slightly perturbed models used in finite difference approximation. A new approach to DSA was thus developed employing finite difference at the primary response level using reanalysis to economically obtain the response of perturbed BEA models

$$\{u\}_{,L} = \frac{\{\tilde{u}\} - \{u\}}{\Delta X_L}; \qquad \{t\}_{,L} = \frac{\{\tilde{t}\} - \{t\}}{\Delta X_L} \tag{16}$$

The simplicity of this approach can be deceiving. After all, differencing two responses is what one might try to obtain sensitivities given no alternatives. The novel ingredient in this context is that iterative reanalysis techniques discussed earlier are employed to obtain the response of the univariately perturbed BEA model in an extremely economical fashion. That is to say, using the reanalysis strategies enables this technique to be considered computationally viable with either the classical implicit differentiation approach or semi-analytical approaches.

A remarkable relationship between the semi-analytical approach and this new approach can be developed starting with

Eq. (14). Substituting Eq. (15) into this expression we obtain

$$[F]\{u\}_{,L} = \left(\frac{[\Delta G]}{\Delta X_L}\right)\{t\} - \left(\frac{[\Delta F]}{\Delta X_L}\right)\{u\} \tag{17}$$

Substituting Eq. (16) into Eq. (17) and multiplying by ΔX_L yields

$$[F]\{\Delta u\} = [\Delta G]\{t\} - [\Delta F]\{u\} = \{v\} \tag{18}$$

Comparison of Eq. (18) with Eqs. (3) and (4) reveals that the computations involved in the semi-analytical DSA approach are equivalent to the first step of simple iteration reanalysis with $\{\Delta u\}^{(0)} = 0$ and $\{\Delta u\}^{(1)} = \{\Delta u\}$. Any additional iterations performed in the reanalysis algorithm serve to improve the accuracy of the prediction for $\{\Delta u\}$, thus making the reanalysis-based UPFD approach superior to semi-analytical DSA. This analysis also helps to explain why this approach is less sensitive to the perturbation step size. In the example problems that follow, this supposition is verified.

Hybrid Approach for Secondary Response Sensitivity

Stress components that are not components of the boundary traction vector (i.e., derived stresses) can be significant elements of an overall response. As described in Refs. 1 and 2, formulae for surface derived stress components can be differentiated to generate a sensitivity formulation. This exact formulation uses $\{u\}_{,L}$ and $\{t\}_{,L}$ obtained via the approach described herein, along with other geometric sensitivity information shown in Refs. 1 and 2, to produce a hybrid sensitivity analysis approach. BIEs are known for displacements and stresses in the interior of BEA models. As described in Refs. 1 and 2, these BIEs can be differentiated to yield new BIEs for the recovery of sensitivities of displacement and stress components that involve nodal displacements and tractions and the sensitivity of these response quantities. Thus, the recovery of internal point sensitivities can also be accomplished via a hybrid approach using the primary response sensitivity $\{u\}_{,L}$ and $\{t\}_{,L}$ obtained via the reanalysis-based UPFD approach described in this paper.

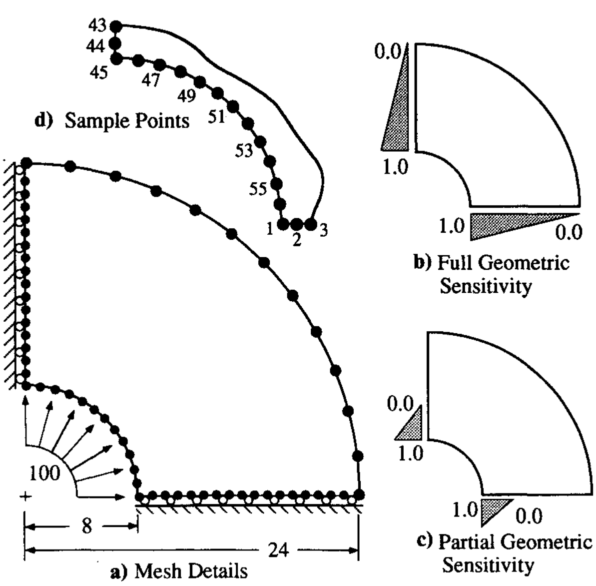

Fig. 1 Hollow cylinder example.

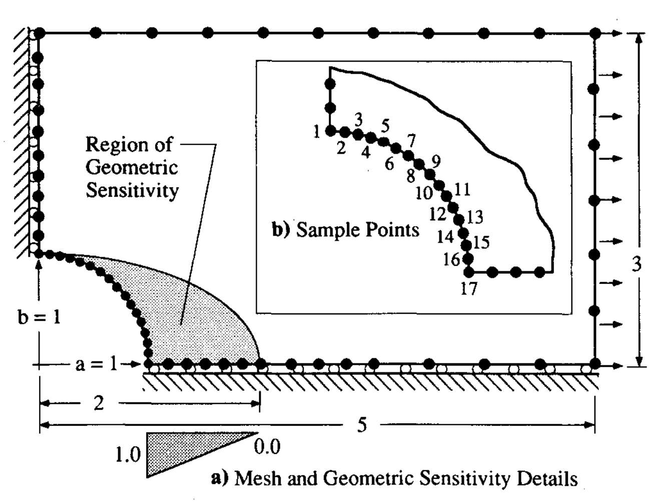

Fig. 2 Plate with hole example.

Table 1a Pressurized cylinder example with full sensitivity

Point	ID approach		FD approach		Hybrid approach	
	$\sigma_{11,L}$	$\sigma_{22,L}$	$\sigma_{11,L}$	$\sigma_{22,L}$	$\sigma_{11,L}$	$\sigma_{22,L}$
1	0.0	7.0987	0.0	7.0313	−0.2674	7.3303
2	−1.6679	8.8029	−1.7968	8.8281	−1.7180	8.7784
3	−3.2325	10.0550	−2.9659	9.9971	−3.3974	10.0430
43	10.0550	−3.1984	9.9971	−2.9659	10.0370	−3.2335
44	8.8029	−1.6679	8.8281	−1.7968	8.7723	−1.7273
45	7.0336	0.0	7.0313	0.0	7.1196	0.0
47	6.5575	0.4729	6.5603	0.4710	6.6213	0.4775
49	5.2691	1.7609	5.2734	1.7578	5.2119	1.7418
51	3.5110	3.5189	3.5156	3.5156	3.5873	3.5952
53	1.7545	5.2770	1.7578	5.2734	1.7318	5.2087
55	0.4690	6.5620	0.4710	6.5603	0.4814	6.7349

Table 1b Plate with elliptical hole example

Point	ID approach		FD approach		Hybrid approach	
	$\sigma_{11,L}$	$\sigma_{22,L}$	$\sigma_{11,L}$	$\sigma_{22,L}$	$\sigma_{11,L}$	$\sigma_{22,L}$
1	−1.8945	0.0	−1.8400	0.0	−1.8840	0.0
3	−1.2685	−0.3040	−1.2700	−0.2990	−1.2684	−0.3040
5	0.2030	−0.7710	0.2000	−0.7580	0.2092	−0.7699
7	1.1584	−0.6655	1.1600	−0.6550	1.1643	−0.6628
9	1.0367	0.0702	1.0510	0.0790	1.0410	0.0744
11	0.3447	0.7063	0.3591	0.7238	0.3468	0.7112
13	−0.1064	0.5962	−0.1020	0.6220	−0.1058	0.6000
15	−0.0945	−0.0464	−0.0978	−0.0300	−0.0947	−0.0501
17	0.0282	−0.4332	0.0160	−0.4500	0.0259	−0.4353

Numerical Results

In the cases presented, Young's modulus $E = 0.3 \times 10^8$ units and Poisson's ratio $\nu = 0.3$ were used. In the tables, ID stands for implicit differentiation, and FD stands for the finite difference method of obtaining sensitivities. Thus, in an FD approach, an analysis and reanalysis are used to obtain stress sensitivities.

Figure 1a shows a quarter symmetry BEA model of a hollow circular cylinder subjected to an internal pressure of 100 units. The model contained 28 three-node quadratic boundary elements and 56 nodes. The original model had an inner radius of 8 units and an outer radius of 24. The inner radius was allowed to vary as shown in Figs. 1b and 1c (i.e., full and partial sensitivities). Stress sensitivities predicted using converged solution vectors for models with an 8.00 and an 8.01 radius are shown in Table 1a for the eight sample point locations shown in Fig. 1d. Reanalysis using simple iteration required only two iterations for convergence. Examination of this table reveals that the accuracy of the sensitivities of the derived stress components predicted via the new hybrid procedure is virtually the same as that obtained by either the direct differentiation of BIEs or from the exact solution. Table 2 shows the breakup of timings for the ID and hybrid approaches. It can be seen that time spent during the reanalysis phase of the hybrid approach can be offset by the savings achieved in the integration phase of the direct method.

Figure 2 displays a quarter symmetry model of a thin strip with a central hole subjected to a unit uniaxial pull. A series of

Table 2 CPU timings for major steps in analysis and DSA for pressurized cylinder example

Step	Full sensitivity ID	Full sensitivity Hybrid	Partial sensitivity ID	Partial sensitivity Hybrid
Analysis				
Preliminaries	3.5	4.1	2.5	2.7
Numerical integration	211.2	211.6	152.1	152.9
Zone assembly	0.5	0.3	0.4	0.4
Overall assembly	10.4	17.4	7.7	15.2
Matrix factorization	100.4	99.5	72.3	74.6
Forward reduction & back substitution	5.7	5.5	4.1	4.4
Surface stress recovery	7.3	7.7	5.5	5.6
Reanalysis				
Preliminaries	0.0	0.0	0.0	0.1
Numerical integration	0.0	197.7	0.0	156.6
Assembly	0.0	0.2	0.0	0.2
Iterations	0.0	21.1	0.0	20.1
Surface stress recovery	0.0	5.7	0.0	5.4
Design sensitivity analysis				
Preliminaries	0.1	0.0	0.1	0.0
Numerical integration	281.5	0.6	151.2	0.5
Zone assembly	2.3	0.0	1.9	0.0
Overall assembly	9.9	0.0	7.4	0.0
Forward reduction & back substitution	9.8	1.2	6.8	1.1
Surface stress recovery	7.7	5.7	5.7	5.7
Total	650.3	578.3	417.7	445.5

sensitivity analyses were performed as the originally circular hole was modified into an elliptical shape of various aspect ratios. Thus, the design variable $a = X_L$ in this problem is the (horizontal) semimajor axis of the elliptical hole. The BEA model was made up of 28 three-node quadratic elements and 56 nodes. This example took five simple iterations to converge during the reanalysis process. As seen in Table 1b, the sensitivities of the derived stress components at the sample points distributed around the hole are in good agreement with the predictions computed using a direct boundary element shape sensitivity analysis of a model with this same geometry.

Conclusions

Shape sensitivity of continuum structural models has been treated by a new hybrid method that does not involve matrix derivatives. This approach represents a highly attractive alternative to the implicit differentiation method in which matrix derivatives are used because of the ease with which it can be implemented in general BEA computer programs. The accuracy and computational efficiency associated with this new method seem to be quite competitive with other methods.

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Dynamic Continuum Plate Representations of Large Thin Lattice Structures

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Introduction

TRUSS-TYPE lattice structures have been the dominant form proposed for large space structures (LSS). Conventional finite element analysis of LSS may require a significant amount of storage capacity and computing time to obtain reliable solutions because of high structural flexibility and large size, especially in the dynamic analysis. Thus, alternative methods have been developed for simplified structural modeling of lattice structures. Of these methods, the continuum methods based on energy equivalence[1] have been shown to give satisfactory results when the wavelength of a vibration mode spans many repeating cells of a lattice structure. The author has proposed a new energy equivalence technique for the large platelike lattice structures,[1] where the idea of using conventional finite element matrices was utilized in the calculation of strain and kinetic energies stored in a repeating cell. The objective of the present Note is to develop classical thin plate continuum models of large platelike lattice structures, where the effects of transverse shear deformations and rotary inertia are negligible. For continuum modeling, three basic assumptions for the lattice plate are made: 1) it behaves grossly as a continuum plate, especially in lower mode vibrations; 2) the ratio of in-plane dimensions to thickness is extremely high; and 3) the deflection is very small compared to the thickness. In this Note, the lattice plate is transformed to the continuum plate by following the same procedure as introduced in Ref. 1. For further discussion, the readers are referred to the figures and nomenclatures used in Ref. 1.

Reduced Equivalent Continuum Stiffness and Mass Matrices for a Lattice Plate

We revisit rectangular lattice plates having different types of repeating cells as shown in Ref. 1. The repeating cells are composed of several different types of lattice elements, and all joints will be considered as nodal points on which nodal displacement vectors are defined. The nodal displacement vector at ith node of (x_i, y_i, z_i) is defined by

$$\{\delta_i\} = \{u_i \quad v_i \quad w_i\}^T \tag{1}$$

Three rotational displacements can be readily added to $\{\delta_i\}$ for the nonhinged-type joints. Neglecting midsurface stretch-

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